Veronika NOVOTNÁ, PhD
Brno University of Technology, Czech Republic
E-mail: novotna@fbm.vutbr.cz
Eng. TerezaVARYŠOVÁ
Brno University of Technology, Czech Republic

SOLVING MACROECONOMIC MODEL USING METHODS OF FUNCTIONAL ANALYSIS

Abstract. *The authors have shown on the original economic problem that the prime solution can be exactly solved by modern mathematical methods and successfully expanded in order to obtain considerably more precise information on the behavior of the model concerned. The expanded model can be and constructed even under more complex assumptions and this allows a more detailed analysis of economic process and add other possibilities to it. In the application part, a specific economic-mathematical model shows construction of such a model and possibilities of its solution. The paper aims to demonstrate how modern methods can be used for solving differential methods with delayed arguments. From the economic point of view, the hypothesis has been confirmed that the studied "shipbuilding cycle", demonstrating the development of the total ship tonnage over time, which is also influenced by the parameter of response's intensity, is stable in terms of small changes in the delay function.*
Keywords**:** *Shipbuilding cycle, delay differential equations, functional differential equation, macroeconomic model, computer simulation.*

JEL Classification: C02, C69

1 Introduction

Economics is a social science which seeks to investigate commercial activities of companies and households, cash flow and production systems in a society. Unlike other scientific fields, it is rather difficult in economics to carry out experiments in a controlled environment and, subsequently, simulate specific conditions in order to test new theories. Economists regard models as tools to help them deal with particular problems; similarly mechanical engineers create models to demonstrate how a specific system works, or biologists use models to illustrate functions of internal organs of living organisms. Models, which are created on the basis of observation of economic reality and acquired statistical data, are based on mathematical disciplines such as numerical methods, statistics, optimization, linear and dynamical programming, operational research, etc. Economic research involves modelling process, analysis, and diagnostics, even solving problems using quantitative methods, as well as explanation and implementation in practice. Therefore appropriate computer software is required.

When modelling complex economic problems, we are faced with a fact that relations between individual quantities change over time. One way to include dynamics of processes in models is to consider time as a continuous quantity and to describe dynamical models using differential equations. The paper explores the possibility of solving a dynamical economic model made up of differential equations with delayed argument. The paper aims to demonstrate how modern methods can be used for solving differential methods with delayed arguments and, subsequently, how these solutions can be presented graphically by means of appropriate software. In the application part, a specific economic-mathematical model shows construction of such a model and possibilities of its solution. Behaviour of the model is demonstrated by computer simulation. The authors present the original model of the "shipbuilding cycle" by Jan Tinbergen and add other possibilities to it. In the paper, the original model is replaced by a new model, which expresses a real economic situation more precisely and even respects a non-constant impact of the history of the factors considered. The equation of the model is then solved using a modern theory of so-called functional differential equations, a special part of which is a theory of linear differential equations with delayed arguments. The paper further demonstrates the possibility of using Maple system for graphical presentation of a solution of the extended model of Tinbergen's "shipbuilding cycle".

2 Modelling in economics

The most prominent economists have their own definitions of the substance of an economic model, some of which are mentioned below.

According to Samuelson and Nordhaus (2009) a model is a formal framework for expressing basic features of a complex system by means of several important relations. They further claim that models are usually presented in the form of mathematical equations or graphs using mathematical software.

Begg, Fischer and Dornbusch(2000) state that a model or a theory creates a series of simplifications which allow a solver to deduce probable behaviour in the future. Hence, it is an intentional simplification of reality.

Economists and mathematicians have long been trying to use dynamic systems in economics. Dynamical processes in economy and the existence of economic cycles were identified by French physicist C. Juglar as early as the first half of the 19th century. All the research paved the way for a new direction in dynamic economics. Half of the 20th century saw an immense development in science and technology, primarily in electrotechnology, mechanics and chemical engineering, but also in biology, ecology, medicine or mathematical economics; references can be found for example in (Andreevaat all., 1992). Economic dynamics was studied by J. Schumpeter (1987), who was mainly interested in economic analysis of dynamics which links economics, history and sociology.

In many real systems which are mathematically modelled by dynamic systems, we are faced with a problem of delayed impact of some quantities modelled. In their monograph, Kobrinskij and Kuzmin(1981) pointed out the necessity of using values of quantities, in dynamic economic models, from the preceding time period, so-called historic quantities, which impact the system development at a given time and lead to changes in the character of the entire process. In his studies, Simonov (2003) modified the existing micro and macro economical models this way and took into account the delay between offer and demand. Another example of adjusting a classical model can be a dynamical Vidal-Wolf's model of single-product sale, which can be found in (Dychta and Camsonjuk, 2003). Currently, some authors adopt Kalecky's model using differential equations with delayed argument (e.g. Collardat all. (2008)).

Study of various model situations, focus on simulating conditions, searching for outcomes, optimal solutions and so on are among the most important current trends. The paper Ioana et al. (2010) presents a new concept for Fuzzy Logic in economic processes. In his paper Zhang (2012), investigated the sensitivity of estimated technical efficiency scores from different methods including stochastic distance function frontier. Many studies have been conducted in order to validate the hysteresis hypothesis, e.g. Alonzo (2011).A New Keynesian model with fiscal and monetary policies interactions is tested for Romanian economy in article Caraiani(2012).

In Mancinis(2003) paper author study a difference partial differential equation, arising from a financial model, whose solution represents the price of a security linked to a dividend-paying stock.

Authors (Plaček, 2013) applied Benford's Law, especially on monthly data of exports and imports of the Czech Republic for the period 1996-2012, which make up the trade balance and use the Z test as the test criterion and the first and second digits are tested.

Tinbergen and his solution of the "shipbuilding cycle" equation

In the 1930s Tinbergen (Tinbergen, 1959) focused on research into economic cycles. In his works he aimed to describe the development of ships' total tonnage over time. It is in this model that a differential equation was used in economics for the first time.

The shipbuilding cycle serves as an example to illustrate the impact of a time delay of some events on the time-consuming process of shipbuilding, building construction, etc.

Let's assume that the total tonnage is a function of time and it will be expressed as $y(t)$. A change in the total tonnage over time shall be $y'(t)$. If in time t the total tonnage is higher than usual, it means that freight rates will be lower and construction of new ships will be gradually terminated (the total tonnage cannot be negative). On the contrary, if in time t the total tonnage is lower than usual, the freight rates will be higher and the shipyard will increase ships' tonnage. An increase or decrease in the total tonnage is dependent on the intensity of shippers' response. For the purpose of further calculations, let's assume that the intensity of the response is constant. Hence, shipbuilding or the rate of change in tonnage is linked to tonnage in time *t-Δ*.

Tinbergen's equation according to (Tinbergen, 1959) is the following:

$$y'(t) = -ay(t - \Delta), a > 0, \quad (1)$$

where*t* is time, *y* is the total tonnage, *Δ* is a delay, a is the intensity of the response, *T* is the end of the period in question. Then, function *y(t)* is the unknown, sought-for function.

Analytical solution of these equations was not and could not be applied in Tinbergen's paper, even with our contemporary knowledge. Tinbergen looked for a solution in interval *[0,T]* not "methodically", but by experimenting.

He assumed that solution y is already given outside interval *[0,T]:*

$$y(t) = h(t), t\epsilon\langle -\Delta, 0) \quad (2)$$

where *h(t)* is a certain given function (let's suppose, for simplification, that the function is continuous). Given the formulation of the problem it makes sense to suppose that *Δ∈[0,T],* while option *Δ≥0* results in a problem without delay and option *Δ=T* results in a problem where the solution is the integral of function *h(t).*

Applying this on the shipbuilding industry, Tinbergen assumes that the level of the total tonnage of new ships is proportional to a delay in variance in tonnage with one-year or two-year delay and a given constant intensity of the system's response.

The so-called initial period determines the relative importance of components. Tinbergen further assumes that the meaning of bigger cycles will be greater if smaller cycles are not distinguishable in the initial period. In this case, periods shorter than two years *Δ<2* connected with the mechanism described can be considered insignificant.

Tinbergen characterizes four cases depending on values of multiplication of parameters a and *Δ*:

- $a\Delta \leq 1/e \approx 0.37$in other words, if the delay and the intensity of the response are small, there won't be any cyclical movement but only a one-sided adjustment to equilibrium y(t)=0, which corresponds with the trend,
- $a\Delta \in (0.37, \pi/2)$we will obtain a damped sinusoid (damped sine wave), i.e. gradual approximation towards equilibrium by steadily decreasing amplitude of the fluctuations,
- $a\Delta = \pi/2$we will obtain a clear sinusoid, i.e. cyclical movement with constant amplitude,
- $a\Delta > \pi/2$we will obtain a sinusoid with an amplitude increasing over time.

Tinbergen's findings correspond to findings of duly derived mathematical statements (e.g. Maruščiak and Olach (2000)).

3 Differential equations with delay

Mathematical models in economics often employ differential equations and their systems, the solutions of which, complying with certain conditions, simulate the behaviour of economic attributes over time t. In general, these are problems concerning solvability, attributes or problems dealing with solution of systems of generally non-linear ordinary differential equations,

$$\begin{gathered} x'_1(t) = f_1\big(t, x_1(t), \dots, x_n(t)\big), \\ \vdots \\ x'_n(t) = f_n\big(t, x_1(t), \dots, x_n(t)\big), \end{gathered} \tag{3}$$

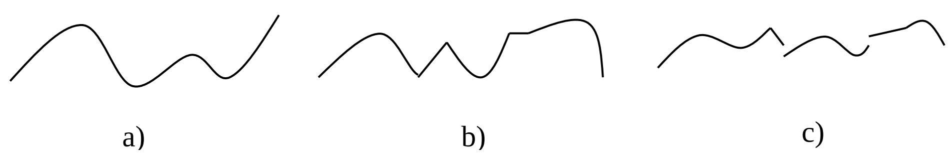
complying with so-called boundary value conditions

$$\begin{gathered} h_1(x_1, \dots, x_n) = 0, \\ \vdots \\ h_n(x_1, \dots, x_n) = 0, \end{gathered} \tag{4}$$

in which functions $f_1, \dots, f_n$and so-called functionals$h_1, \dots, h_n$meet justified requirements in the theory of such problems. At the same time, preconditions of the given problems determine the attributes of the problem's solution (3), (4) ranging from "classic" (all components are continuous or continuously differentiable functions of independent variable t), via Carathéodory's solution in the first half of the 20th century (components of the solution are continuous but its derivation does not have to necessarily exist in the "zero level set"), to a so-called generalized solution whose components can be only partly Carathéodory's ones (studied since the second half of the 20th century).

To clarify – the graph of the classic solution is a "smooth" curve (a tangent can be drawn in each point of the curve), the graph of Carathéodory's solutions is a continuous curve with points where a tangent cannot be drawn (points, tips) and the graph of the generalized solution is a curve made up of "broken" curves of the previous types.

Figure 1.a) Classic sol., b) Carathodory's sol., c) Generalized sol. Source: own



By employing vector specification $x = (x_i)_{i=1}^n, f = (f_i)_{i=1}^n$, a $h = (h_i)_{i=1}^n$ problems (3) and (4) can also be written as

$$x'(t) = f\big(t, x(t)\big), h(x) = 0 \tag{5}$$

As the theory of differential equations states, every higher-order differential equation can be represented by an equivalent system (3) and boundary conditions for the solution in question in form (4). At the same time, studying problems for higher-order differential equations (systems) in the initial form may lead to an "easier" and "more detailed" solution's description. Problems (3), (4) will be considered for our purposes. It should be mentioned that special examples of system (3) are linear and quasilinear systems and special examples of boundary conditions (4) are so-called initial conditions.

$$(x_1(t_{n0}))_{i=1}^n = (c_i)_{i=1}^n,$$

resp.

$$x(t_0) = c, \tag{6}$$

where $c = (c_i)_{i=1}^n$ is a vector of real constants.

Mathematical models used in order to describe economic processes have long used and still extensively use the "classic" model of problem (3), (4). As new methods of "Carathéodory’s theory" of differential equations developed and the general theory of so-called functional differential equations emerged and developed, new mathematical models taking into account even the real economic relationships have been emerging since the second half of the 20th century. Let us now analyze the above mentioned Tinbergen's problem using the modern theory of so-called functional differential equations, a very special part of which is also the theory of linear differential equations with delayed argument. An extensive and complex analysis of such equations can be found mainly in (Azbelev, 2003).

Available literature dealing with solvability of systems of differential equations with delayed arguments, which can be applied also in economic practice, includes a wide range of practical and useful findings. Conditions of solvability (i.e. existence and unambiguity) of given problems, both general and special, are subject to detailed scrutiny, so are conditions for their correctness (i.e. little dependence of a solution on "small" changes in initial conditions and parameters required for a numerical solution), conditions for nonnegativeness of a solution and others, including description of the method used to obtain a solution.

The general theory, which makes it possible to solve not only the above mentioned problems, but also others, can be found, for linear example, in monograph (Kiguradze and Půža, 2003). The theory for a non-linear example was covered in studies published in magazines by both authors. An application on the above mentioned types of differential equations with delay, including the description of the way the desired solution was designed can be found in for example (Kuchyňková and Maňásek, 2006) and the cited bibliography.

Although solutions of specific differential equations with delay were first mentioned several centuries ago, it was not until the middle of the 20th century that a systematic theory of such equations was developed. It is reflected in the procedures used to obtain solutions of these equations applied in practice. First methods were based on an analogy with solutions of classic differential equations (the Euler method, Runge-Kutta etc.). However, due to their "localness" those methods are not regarded as appropriate for solving "global" problems of functional differential equations. An adequate procedure is to apply a fixed-point theorem, which corresponds to the concept of the entire theory of problems for functional differential equations and which is also applied in this paper.

4 Extension of Tinbergen's equation and its solution employing modern methods

Thanks to the current range of possibilities provided by functional analysis and the theory of differential equations with delay, a method is available presently which can be used to solve a considerably more general mathematical model, providing us with a wider range of findings, including comparison of various parameters' impact.

Let us focus on a problem stemming from the original Tinbergen's equation:

$$y'(t) = -a(t)y\big(t - \delta(t)\big), t \in [0, T] \tag{7}$$

$$y(t) = h(t), t \in [-\tilde{\Delta}, 0]. \tag{8}$$

where functions *a* and *δ* are (for simplification) continuous on interval *[0,T], δ(t) ≥0* for $t \in [0, T], \tilde{\Delta} = \min\limits_{t\in[0,T]}(t - \delta(t))$and *h* is a continuous function on interval $[-\tilde{\Delta}, 0]$. If $\delta(t) \equiv \delta \geq 0$(i.e. the invariable), then the equation (7) will be an equation with constant delay and $\tilde{\Delta} = \Delta$. If, in addition, *a(t)≡a>0* (i.e. constant), it is a differential equation with constant delay explored by Tinbergen.

In compliance with the theory of boundary value problems of functional and differential equations quoted in the introduction, equation (7) seems to be a first-order linear differential equation with generally variable coefficient and non-constant delay. As functions *δ* and *h* are assumed to be continuous, the solution of such an equation on interval *[0,T]* is a continuously differentiable equation (a "classic" solution). It follows from the general theory of boundary value problems of functional differential equations [15] that every initial problem, i.e. solving an equation (7) complying with the initial condition (6) in form *x(0)=c* (where c is a vector of real constants), is positively solvable and the problem is correct (i.e. "small" changes in input data only lead to "small" changes in the solution). Therefore it makes sense to employ numerical methods in order to obtain an existing and the only possible solution. We shall use the procedure justified for solving a certain category of boundary (value) problems of functional differential equations in (Maňásek, 2007). Specifically his Theorem 4.2.for solving a system of functional differential equations in vector format.

$$x'(t) = p(x)(t) + f(x)(t),\ t \in [0, T] \tag{9}$$

with an initial condition

$$x(0) = c \tag{10}$$

follows.

Theorem 1

Let $\rho > 0$ that for all $\lambda \epsilon [0,1]$and arbitrary solution x of the problem

$$x'(t) = p(x)(t) + \lambda f(x)(t), t \in [0, T], x(0) = \lambda c$$

satisfies the estimate$\|x\|_c \leq \rho$.

Then the problem (9), (10) has at last one solution and for any $x_0$continuous$t \in [0, T]$there are sequences $\{x_m\}_{m=1}^{\infty}$ of the solution of the problem

$$x'_m(t) = p(x_m)(t) + f(x_{m-1})(t),$$
$$t \in [0, T], x_m(0) = c \tag{11}$$

such that x for which $\|x - x_m\|_c \to 0$ for $m \to \infty$ is a solution of the problem (9), (10).

In this assertion, p and f are so-called operators, specific examples of which are right sides of systems of ordinary differential equations, equations with delay and others, and ‖.‖_c is a norm in the space of continuous (vector) functions.

This assertion valid for one linear differential equation with one delay leads to an assertion mentioned below concerning its solution, i.e. an assertion obtained by selecting operators *p* and *f* (listed in Theorem 1).

In this paper, in order to solve equations (7), we shall use the method of gradual approximations based on the fixed-point theorem of the operator determined by an integral form of the initial problem. Specifically in this case, we are seeking a solution of problems (7) and (8), i.e. a solution of a first-order ordinary linear differential equation (7), with delay δ(t), initial condition *y(0)=h(0)* and „historic function“ h defined on interval $[-\tilde{\Delta}, 0]$, as a limit of arithmetic progression $\{y_n\}_{n=1}^{\infty}$of solution of problem without delay:

$$y'_n(t) = -a(t) y_{n-1}\big(t - \delta(t)\big), t \in [0, T] \tag{12}$$

$$y_n(t) = h(t), t \in [-\tilde{\Delta}, 0] \tag{13}$$

while an initial function y_0 (t) can be a random function continuous on interval *[0,T]* and starting from point $[0, h(0)]$.

In order to describe the below mentioned method of solution of problem (7), (8) we shall employ the following function defined on interval *[0,T]:*

$$\chi_{[0,T]}\big(t-\delta(t)\big)=\begin{cases}1\ if\ \big(t-\delta(t)\big)\in[0,T]\\0\ if\ \big(t-\delta(t)\big)\notin[0,T].\end{cases}\tag{14}$$

It enables us to write problem (7), (8) down the following way:

$$\begin{aligned}y'(t)&=-a(t)\chi_{[0,T]}\big(t-\delta(t)\big)y\big(t-\delta(t)\big)\\&-a(t)\Big(1-\chi_{[0,T]}\big(t-\delta(t)\big)\Big)h\big(t-\delta(t)\big),t\in[0,T]\end{aligned}\tag{15}$$

$$y(0)=h(0),\tag{16}$$

Theorem 2

Let functions a and δ be continuous on interval *[0,T]* and function h continuous on interval $\tilde{\Delta}$. Then the initial problem (7), (8) has got only one solution y and for a random function y_0 continuous on interval *[0,T]* and each $n\in N$ there is a progression $\{y_n\}_{n=1}^{\infty}$ of problem solutions

$$y'_n(t)=-a(t)\chi_{[0,T]}\big(t-\delta(t)\big)y_{n-1}\big(t-\delta(t)\big)$$
$$-a(t)\Big(1-\chi_{[0,T]}\big(t-\delta(t)\big)\Big)h\big(t-\delta(t)\big),t\in[0,T],$$
$$y(0)=h(0),$$

so that $y(t)=\lim_{n\to\infty}y_n(t)$ on interval $[0,T]$ evenly.

In this paper we shall focus on the fact that in his study Tinbergen describes a solution of equation (1) without considering „historic development " over $t<0$ or the fact that the length of a delay does not have be constant. If we proceed from the assumption that known values from the past can be interpolated with a suitable function, then a modern theory of differential equations with delay can be used and the problem can be solved while taking into account even these facts.

5 Graphical solution of a model

5.1 Maple system

There are a lot of programs designed for solving complex mathematical problems. Not only do more advanced mathematical programs enable solutions of equations and work with algebraic formulae, but they can also be used for integral and differential calculus, drawing graphs, work with units, interactive change in documents, numerical calculus, etc.

Our requirement was that software should allow comfortable solutions of differential equations and graphical presentation of findings. The authors of the paper could choose between the R language, Matlab and Maple system.

R language (or R project), a free software programming language, is a language and software environment primarily designed for statistical computing. It allows relatively easy programming of other applications including graphical output.

Matlab is a high-level language intended for technical calculations and an interactive environment for development of algorithms, data visualization and analysis, and for numerical computation. Programming language Matlab allows fasters solutions of technical computation than common programming languages such as C, C++ and Fortran.

Maple can be described as a multi-purpose mathematical software tool. It incorporates a high-performance mathematical computing core with fully integrated numerical computation and symbolic computation available from WYSIWYG environment of a document. Maple environment secures fast and available quantification, visualization, animation and simulation of real phenomena.

It was thanks to a wide range of possibilities provided by Maple that the authors of the paper decide to use it to solve a model.

Maple offers a function intended to solve differential equations using a method of steps. Due to limitations of the method the function was not used, but the above mentioned theories were transferred to a solution of differential equations with delayed argument, which allows the authors to employ common numerical procedures of solving ordinary differential equations available in Maple.

Graphical solution took advantage of the possibilities of "library plots", which enabled graphical presentation of obtained solutions.

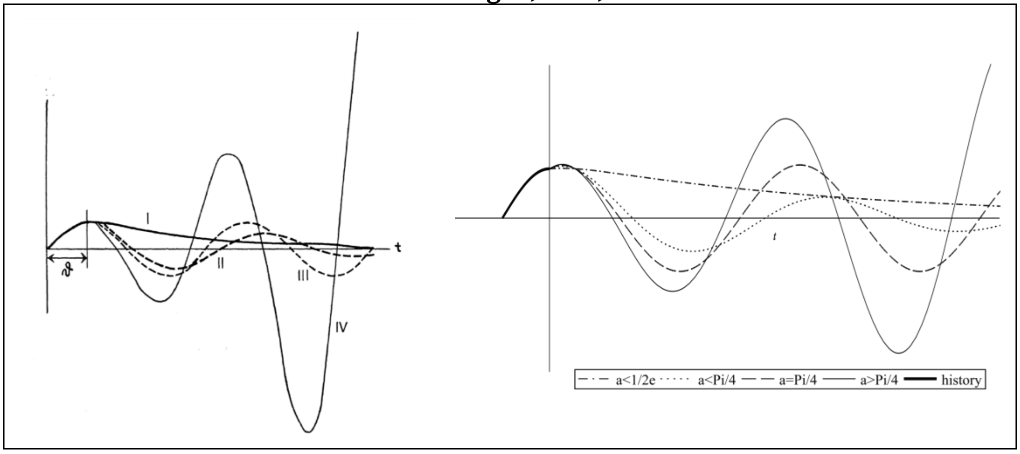
5.2 Solution under original initial conditions

In order to demonstrate possibilities of a new approach towards solution of the original problem, let's suppose that "historic development" before moment $t=0$ can be simulated by function $\mathrm{y} = \cos(\mathrm{t})$The same (constant) time delay and representation in one graph was used to allow easier comparison of particular results for various values of parameter a. Parameter a was set based on Tinbergen's findings $\mathrm{a}\Delta < 1/e$, $1/\mathrm{e} < a\Delta < \pi/2$, $\mathrm{a}\Delta = \pi/2$ and $\mathrm{a}\Delta > \pi/2$.

Specifically, the parameter shall be $\Delta = 2$(which is the value that Tinbergen started with).

The outcome of our solution, shown in Fig. 2., clearly corresponds to Tinbergen's findings. It is further supported by comparison of initial Tinbergen's drawing and the Maple output.

Figure 2.ComparisonofTinbergen's model and Maple output. Source: Tinbergen,1959;ownwork

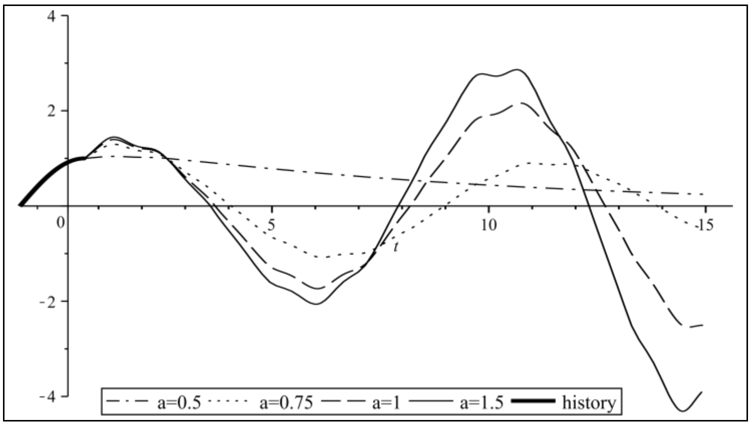


5.3 Solution with variable delay

Let's proceed from the assumption that delay *δ(t)* is not constant but it is a function of time *t*. The assumption of a constant delay is artificial, in reality the length of a delay will fluctuate. This is mentioned by Arlt and Radkovský(2001), who came to the conclusion that constant characteristics of a delay cannot be expected during practical analysis of a delay of economic time series.

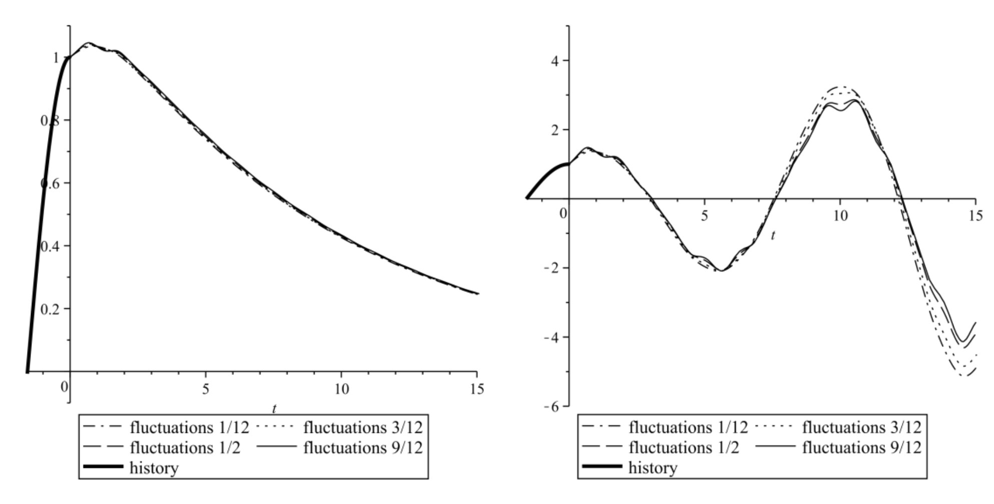
For example, let's assume that a delay can be expressed by function $\delta(\mathrm{t}) = 2{,}5 + 0{,}5\sin(5\mathrm{t})$. Hence $\delta(\mathrm{t}) \in [2;3]$for $t \in [0, T]$,i.e. the delay will fluctuate around 2.5 with amplitude 0.5 and it will be fivefold faster than a business cycle. Parameter a shall be gradually set according to Tinbergen's findings.

Figure 3. Model with non-constant delay. Source: own work.



As follows from the comparison between the model with a constant delay and the model with the above selected delay dependant on time t (Fig. 3.), the solutions are similar to a certain extent, but the non-constantness of a delay is reflected in a slight distortion of the solution. The selected delay only had 20 % amplitude unlike the constant delay, and the non-constantness of a delay is shown primarily in the initial period which is fully influenced by the past.

Figure 4.Cases I and IV withdifferentdelayfluctuations. Source: ownwork.



As Fig. 4.shows, periodical function of delay *δ(t)* (simulating fluctuations usual in economic time series), causes only slight distortions of our model unlike the constant delay with function value *δ(t).* The figure shows graphs for cases I and IV and some cases for "fluctuating" delay. Such cases were selected which are still real in economic terms. From the economic point of view, the hypothesis has been confirmed that the studied "shipbuilding cycle", demonstrating the development of the total ship tonnage over time, which is also influenced by the parameter of response's intensity, is stable in terms of small changes in the delay function.

If the amplitude of delay exceeds limits set for the model with constant delay, the impact of such delay will be more significant, i.e. assumptions I-IV concerning the value of multiplication *aδ(t)* will not be complied with.

6 Conclusion

Modelling dynamical models is not an autotelic process. It primarily aims to obtain tools to analyze the behaviour of complex systems changing over time in which experiments, for practical reasons, cannot be carried out. Unlike physics, where at least some processes can be verified experimentally, such method is not viable in economic processes. Therefore, modelling such processes in as detailed way as possible is one of few ways to study the behaviour of economic systems under initial conditions.

In this paper, we have pointed out the possibility of employing the modern theory of differential equations with delay in economic modelling. We have shown on the original Tinbergen's problem that the original solution can be exactly solved by modern mathematical methods and successfully expanded in order to obtain considerably more precise information on the behaviour of the model concerned. As was presented in the paper, the expanded model can be solved (under different conditions) and constructed even under more complex assumptions or possibly monitor the impact of particular parameters of a model on its solution. All this allows a more detailed analysis of economic process.

Specific results have been presented as an example and thanks to computational processing of the problem the results can be presented graphically.

Acknowledgment

This paper was supported by grant FP-S-15-2787 'Effective Use of ICT and Quantitative Methods for Business Processes Optimization'' from the Internal Grant Agency at Brno University of Technology.

REFERENCES

[1] **Alonzo, P. and Di Sanzo, S.(2011)**, ***Unemployment and Hysteresis: A Nonlinear Unobserved Components Approach;*** Studies in Nonlinear Dynamics and Econometrics, Vol.15(2);
[2] **Andreeva, E. A.,Kolmanovskij, V. B. and Sajchet, L. E. (1992), *Upravleniesistemami c posledejstvem*,** Moskva: Nauka**.**
[3] **Arlt, J. andRadkovský, Š. (2001), *The Lag Analysis in Modelling of Relationship of Economic Time Series;*** Politickáekonomie, vol. 49, No. 1, pp. 58-73;
[4] **Azbelev, N.V.(2003)**, ***Kaketobylo (obosnovnychetapachpazvitijasovremnnojteoriifunkcionalno-differencialnychupravnenij)***,Problemynelinejnnogoanalyza v inženernychsistemach, Vol.9 (17);
[5] **Begg,D. , Fisher, S. and Dornbush, R. (2000), *Economics*;** *McGraw-Hill;*
[6] **Caraiani, P.(2012)**, ***Monetary and Fiscal Policies Interactions in an Estimated New Keynesian Model for Romania***. *Economic Computation and Economic Cybernetics Studies and Research,* Vol. 1; ASE Publishing;
[7] **Collard, F., Licandro, O. and Puch, L. (2008),*The Short Run Dynamics of Optimal Growth Models with Delays*,** Annales d'Economie et Statistique, vol. 90, pp. 127-144;
[8] **Dychta,B.A., Camsonjuk, O.N. (2003),*Optimalnoeimpulsnoeupravlenie c priloženijami*.** Moskva: Fizmatlit;

[9] **Ioana, A., Mirea, V., Bălescu, C. (2010),*Economic Processes Study through Fuzzy Logic*.** *Economic Computation and Economic Cybernetics Studies and Research,* Vol. 2; *ASE Publishing*;
[10] **Kiguradze, I., Půža, B. (2003),*Boundary Value Problems for Systems of Linear Functional Differential Equations*;** Folia Fac.Sci.Mat. MasarykianaeBrunensis, Mathematica, vol. 12, Brno
[11] **Kobrinskij, N.E., Kuzmin, V.I. (1981), *Točnostekonomiko-matematičeskichmodelej*,** Moskva: Progress;
[12] **Kuchyňková, M., Maňásek, L. (2006), *On Constructing a Solution of a Multi-point Boundary Value Problem for the Pantograph Equation***; Folia Fac.Sci.Mat.MasarykianaeBrunensis, Mathematica, Vol. 16;
[13] **Maplesoft, (2012), *Math Software for Engineers, Educators & Students,*** Available from: http://www.maplesoft.com;
[14] **Manasek, L. (2007), *On Constructing a Solution of a Boundary Value Problem for Functional Differential Equations*;** Proceedings of Equadiff, vol. 11, pp. 237-244**;**
[15] **Mancini, C. (2003), *Uniqueness of the Solution to a Difference-Partial Differential Equation for Finance*;** Mathematical Models and Methods in Applied Sciences, vol. 13, no. 7, pp. 919-943;
[16] **Marušiak, P. andOlach,R. (2000), *Funkcionalnediferencialnerovnice,*** Zilina: EDIS;
[17] **Placek,M. (2013),*Utilization of Benford's Law by Testing Government Macroeconomics Data;*** European Financial Systems 2013, pp. 258-264**;**
[18] **Samuelson, P. A. and Nordhaus,W. D. (2009), *Economics*;** *Irwin/McGraw-Hill;*
[19] **Schumpeter,J. A. (1978), *Teoriahospodarskehovyvoja (from germanorig: Theorie der wirtschaftlichenEntwicklung*),** Bratislava: Pravda**;**
[20] **Shampine, L. F. and Thompson,S. (2001),*Solving ddes in Matlab*;** Applied Numerical Mathematics, vol. 37, no. 4, 441-458;
[21] **Simonov,P. M. (2003), *Issledovanieustoichivostirešenijnĕkotorychdinamičeskichmodelyachmikro - imakroekonomiki*,** Vestnik PGTU 2003**;**
[22] **Tinbergen,J.(1959), *A Shipbuilding Cycle?*,**Klaasen, L. H., L. M. Koyck and Witteween H. J. (eds.) Jan Tinbergen - selected papers, *North-Holland Publishing Company,* Amsterdam**;**
[23] **Zhang, T. (2012), *A Monte Carlo Analysis for Stochastic Distance Function Frontier*,** Inzinerine Ekonomika-Engineering Economics**, 23(3), 250-255.**